The Silmarillion
13 October 2020

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- Arrows: nondecreasing monotonic functions

Remark. Every arrow $[n] \to [m]$ can factor into a unique composition of surjective degeneracy maps followed by injective face maps.

Face maps: $d_i^n : [n-1] \to [n]$; the image misses i

Degeneracy maps: $s_i^n : [n+1] \to [n]$; i and $i+1$ in $[n+1]$ both map to $i \in [n]$

$$d_i^n(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geq i \end{cases} \qquad s_i^n(x) = \begin{cases} x & \text{if } x \leq i \\ x-1 & \text{if } x > i \end{cases}$$

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$$[3] \xrightarrow{s_0^2} [2]$$

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$$\begin{array}{ccccccc} [3] & \xrightarrow{s_0^2} & [2] & \xrightarrow{d_0^3} & [3] & \xrightarrow{d_2^4} & [4] \\ 0 & & 0 & & 1 & & 1 \\ 1 & & 0 & & 1 & & 1 \\ 2 & & 1 & & 2 & & 3 \\ 3 & & 2 & & 3 & & 4 \end{array}$$

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0	0	1	1	1
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2	1	2	3	3
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It suffices to only think about d_i^n and s_i^n, $i \in \{0, \ldots, n\}$.

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Write $X_\bullet$ for a simplicial object.

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- Arrows: natural transformations $X_\bullet \to Y_\bullet$, i.e., a family of arrows $f_n : X_n \to Y_n$ that commute with maps $X_n \to X_m$ and $Y_n \to Y_m$. By before, it is enough to commute with face and degeneracy maps.

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\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
{\scriptstyle \partial_i^n}\downarrow\uparrow{\scriptstyle \sigma_i^{n-1}} & & {\scriptstyle \partial_i^n}\downarrow\uparrow{\scriptstyle \sigma_i^{n-1}} \\
X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1}
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$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \partial_i^n \downarrow \uparrow \sigma_i^{n-1} & & \partial_i^n \downarrow \uparrow \sigma_i^{n-1} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array}$$

Maps of simplicial objects are just degreewise maps that commute with face and degeneracy maps.

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- $\partial_i^{n+1}\sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1}\partial_i^n & \text{if } i < j; \\ \mathrm{id}_{X_n} & \text{if } i = j \text{ or } i = j+1; \\ \sigma_j^{n-1}\partial_{i-1}^n & \text{if } i > j+1. \end{cases}$

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- the n-simplex.

  $$X_n = \Delta^n = \{(t_0, \ldots, t_n) \subseteq \mathbf{R}^{n+1} \mid 0 \leq t_i \leq 1 \text{ and } \sum t_i = 1\}$$

  and ∂_i^n, σ_i^n are the geometric face and degeneracy maps. ∂_i^{n-1} includes the ith face, σ_i^{n+1} projects onto the ith face. (This is a co-simplicial object.)

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- the "classifying space" (not quite yet). Let G be a group. Let $X_n = G^n$ and

  $$\sigma_i(g_1, \dots, g_n) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n)$$

  $$\partial_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0; \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n; \\ (g_1, \dots, g_{n-1}) & \text{if } i = n. \end{cases}$$

Simplicial objects

Geometric realization. Let $X_\bullet$ be a simplicial set; i.e., a functor $\Delta^{op} \to \mathbf{Set}$; i.e., X_n is a set and ∂_i^n, σ_i^n are set maps. We can construct a topological space $|X_\bullet|$.

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1. $X_n \times \Delta^n$ is $|X_n|$-many disjoint copies of the topological n-simplex.
2. Take their disjoint union $\coprod X_n \times \Delta^n$.
3. Glue $(x, s) \in X_m \times \Delta^m$ to $(y, t) \in X_n \times \Delta^n$ if there is a map $[m] \to [n]$ inducing $X_n \to X_m$ and $\Delta^m \to \Delta^n$ such that $y \mapsto x$ and $s \mapsto t$.

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n-simplexes of the form $\{\sigma_i^{n-1}(y)\} \times \Delta^n$ for some $y \in X_{n-1}$ are degenerate. We say $\sigma_i^{n-1}(y) \in X_n$ is degenerate and nondegenerate otherwise; nondegenerate elements index the n-cells of $|X_\bullet|$, a CW complex.

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Adjointness: $\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{s\mathbf{Set}}(X, SY)$.

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This is a chain complex; the simplicial identity condition that $\partial_i^{n-1}\partial_j^n = \partial_{j-1}^{n-1}\partial_i^n$ if $i < j$ implies $d^2 = 0$.

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This is a chain complex; it is a subcomplex of the unnormalized complex.

Theorem. The homologies of the unnormalized and normalized chain complexes are isomorphic.

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This theorem is a small piece of a bigger picture.

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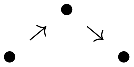

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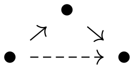

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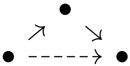

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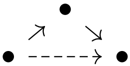

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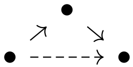

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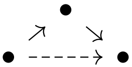

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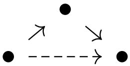

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- Model categorically: to come.

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- Consequently, simplicial abelian groups, simplicial R-modules, simplicial rings, simplicial k-algebras, simplicial fields (which are boring) are all Kan complexes.
- Other examples do exist; the simplicial set BG is a Kan complex for all G, but a simplicial group if and only if G is abelian.

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Remark. The horn filling condition makes $\sim$ an equivalence relation and ensures $\pi_n X_\bullet$ is well-defined.

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- $\pi_n -$ realizes $|BG|$ as a $K(G,1)$ space; i.e., $\pi_1 BG = G$ and $\pi_{n \neq 1} BG = \{1\}$.
- Given a simplicial group $X_\bullet$, $\pi_n X_\bullet$ is independent of $*$. We typically choose $* = 0 \in X_0$.

Big piece. Dold-Kan correspondence.

Simplicial objects

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If $\mathcal{A}$ is an abelian category, then there is an equivalence of categories

$$s\mathcal{A} \xrightarrow{N} \mathbf{Ch}_{\geq 0}\mathcal{A}$$

{simplicial objects in $\mathcal{A}$}	{connective chain complexes in $\mathcal{A}$}

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$KC_\bullet$ (face maps only):

$$C_0 \leftleftarrows C_0 \oplus C_1 \; \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \; C_0 \oplus (C_1)^2 \oplus C_2 \; \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \; C_0 \oplus (C_1)^3 \oplus (C_2)^3 \oplus C_3 \cdots$$

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N and K are an adjoint pair:
$\mathrm{Hom}_{s\mathcal{A}}(KC_\bullet, X_\bullet) \cong \mathrm{Hom}_{\mathbf{Ch}_{\geq 0}\mathcal{A}}(C_\bullet, NX_\bullet)$

∞-categories

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- The compositions/factorizations of $\mathcal{C}$ are the elements in $N\mathcal{C}_n$.

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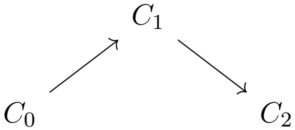

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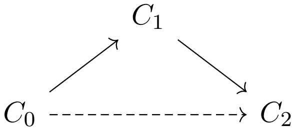

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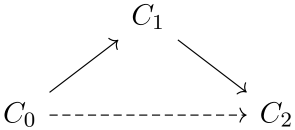

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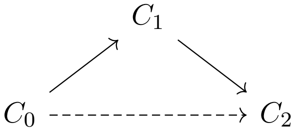

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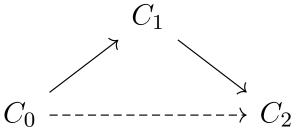

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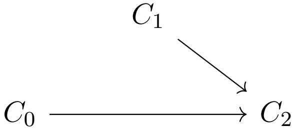

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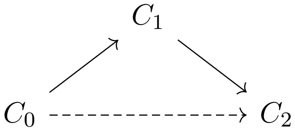

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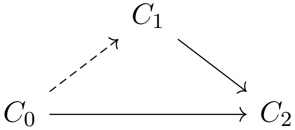

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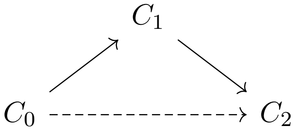

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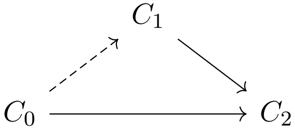

This exists if $\mathcal{C}$ is a groupoid.

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Theorem. Every ∞-category is realized as the nerve of some small category $\mathcal{C}$.

Remark. Of course, Kan complexes are ∞-categories, since all horns are filled, not just inner horns. Kan complexes are ∞-groupoids. (Also $(\infty, 0)$-categories)

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An (∞, n)-category is a category with higher morphisms and the k-morphisms are invertible for $k > n$.

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- The construction of model categories is topologically motivated, and very similar to Waldhausen categories to come.

Model categories

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closed under composition and retracts and subject to the axioms

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$$X \xrightarrow{(c)(w)} Z \xrightarrow{(f)} Y \text{ and } X \xrightarrow{(c)} Z' \xrightarrow{(f)(w)} Y;$$

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Definition. Dually, we can factor $X \to *$ as

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Remark. Given a model category $\mathcal{M}$, its opposite $\mathcal{M}^{op}$ is also a model category where

weak equivalences correspond to their opposites,

fibrations are opposites of cofibrations, and

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3. Also notice that every object is cofibrant, since $\emptyset \to X_\bullet$ is a levelwise injection.

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2. Fibrant objects are ∞-categories; $X_\bullet \to *$ is a fibration if $X_\bullet$ is an ∞-category.

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   - Cofibrations are determined.
2. This has the benefit of carrying homotopical information, since every object will be fibrant, thus we can take homotopy, and weak equivalences are homotopy equivalences. These model categories give a setting for homotopical algebra.

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Remark. Up to weak equivalence in the Dwyer-Kan model category structure of simplicial categories, every simplicial category is the localization of a category with weak equivalences.

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   - There are many: **Lemma.** If X is an object, then $(X^f)^c$ and $(X^c)^f$ are fibrant and cofibrant.

Proof (because we should do at least one).

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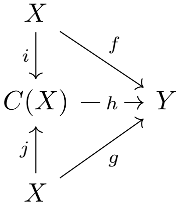
   - Define a (left) homotopy between f, g if there exists a diagram



such that there exists $p : C(X) \xrightarrow{(w)(f)} X$ such that $pi = pj = \mathrm{id}$ and $i \sqcup j : X \sqcup X \xrightarrow{(c)} C(X)$.

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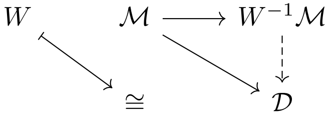

5. **(Hard) Theorem.** [Quillen] $W^{-1}\mathcal{M} \cong \mathcal{M}^{cf}/\sim$.

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Simplicial rings are dense in connective spectra. For treatment of this, see section 3.3 [Dundas, Goodwillie, McCarthy].